Distorted-Wave Born Expansions*

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A successive approximation procedure is presented for calculating transition amplitudes for direct and rearrangement collision processes. The essential feature of the method is that the base problem about which the solution is expanded exhibits all the bound states (of subsystems of particles) which appear in initial or final scattering states. While no proof is given that our expansion converges, arguments which have been presented indicating that the ordinary Born expansion diverges for a wide class of problems no longer apply to the expansion proposed here. A particular choice of base problem leads directly to the well-known distorted-wave Born approximation. The variational nature of this approximation is exhibited and, as a consequence of the general formalism, a procedure for systematic improvement is presented. Circumstances are described under which the first term in the modified Born expansion has an error of known sign. The use of separable potentials to generate solutions to the base problem is discussed and is shown, in the three-body case, to lead to a model proposed recently by Amado. As a by-product of our work a variational principle for transition amplitudes is developed which is a generalization of the Kohn principle for the two-body elastic amplitude and is valid for any scattering process described by the Schrödinger equation.

1. INTRODUCTION

THERE has been an increasing amount of attention turned recently toward the development of perturbational techniques for scattering problems when the ordinary Born expansion fails to converge.[1–3] Since this failure is intimately connected with the presence of bound states the need for new techniques is particularly apparent in the treatment of rearrangement collisions,[4] where the presence of interaction potentials which are strong enough to form bound states is an intrinsic feature of the problem. Actually, the divergence of the Born expansion is more directly related to poles in the complex λ plane, where λ is the potential strength parameter, rather than to poles in the complex energy plane, and a rigorous mathematical study of the problem must take this distinction into account.[2] Our aims here are more modest. We present a modification of the Born expansion which at least avoids the Aaron-Amado-Lee criticism.[4] No convergence proofs are offered. We rely solely on the expectation that if a potential is too weak to cause binding an expansion about that potential will converge over a wide range of energies. The possibility that the potential may still be strong enough to cause a resonance (a pole in the complex λ plane) is recognized but not discussed further here.

The basis of our method is an identity developed by Kato[5,6] for two-body scattering in a particular partial wave and here extended to deal with transition matrix elements in general [see Eq. (2.23)]. An essential feature of the Kato identity, which is maintained in the generalization developed here, is that one term in the identity is explicitly of second order so that its neglect gives rise to a variational principle for the transition matrix elements. The distorted-wave Born expansion is derived in Sec. 3 for two-body scattering. This formulation may be viewed as an alternative to Weinberg's method of quasiparticles.[2] A generalization of the formalism is discussed in Sec. 4, along with two applications. We first consider a three-body scattering problem and construct a solvable base problem by making a particular choice of separable potentials to replace the true potentials. This base problem turns out to be identical to a model proposed recently by Amado.[7] We are therefore able to justify Amado's model in the framework of ordinary potential scattering theory and, at the same time, to give a procedure for systematic improvement of this first approximation. As a second application we show that the well-known stripping approximation[8] [e.g., for a (d,p) reaction] can by a suitable choice of distorted wave be made to appear as the first term in a distorted-wave Born expansion. We have therefore exhibited the variational nature of the stripping (or distorted-wave Born) approximation, and this is perhaps the best way to understand its success. This success in turn may be taken as some indication that the expansions discussed here will be of practical utility in a variety of applications.

Of the two applications discussed in Sec. 4 one is specifically a three-body problem while the other, the stripping example, is effectively reduced to a three-body problem by the introduction of optical potentials. It may well be that the requirement of a solvable comparison problem limits the domain of applicability of the method proposed here to two- and three-body models. This point does, however, deserve further investigation;

* Supported by the National Science Foundation.
[1] S. Tani, Phys. Rev. **117**, 252 (1960).
[2] S. Weinberg, Phys. Rev. **130**, 776 (1963); **131**, 440 (1963); **133**, B232 (1964).
[3] M. Rotenberg, Ann. Phys. (N. Y.) **21**, 579 (1963).
[4] R. Aaron, R. D. Amado, and B. W. Lee, Phys. Rev. **121**, 319 (1961).
[5] T. Kato, Progr. Theoret. Phys. (Kyoto) **6**, 394 (1951).
[6] L. Rosenberg and L. Spruch, Phys. Rev. **125**, 1407 (1962). The Kato identity was generalized here to cover binary collision processes, the system having a well-defined total angular momentum. A minimum principle for elements of the K matrix was then constructed. This paper contains references to earlier work by the authors along similar lines.
[7] R. D. Amado, Phys. Rev. **132**, 485 (1963).
[8] S. T. Butler, *Nuclear Stripping Reactions* (John Wiley & Sons, Inc., New York, 1957).

the shell model, e.g., might serve as a useful comparison problem for nucleon–nucleus scattering.[9] In any event, we have chosen to use a many-body model in the derivation of the basic identity given in Sec. 2. We believe this identity, and the variational principle which follows directly from it, to be of general interest, independent of the particular applications discussed here.

2. THE BASIC IDENTITY

The multichannel scattering problem under consideration is defined by the Schrödinger equation

$$(H-E)\Psi_\alpha^{(\pm)}=0 \tag{2.1}$$

with boundary conditions which specify the particular entrance channel and whether the scattered wave is outgoing (+) or incoming (−) at infinity. In order to write down these boundary conditions in a convenient form we must introduce some notation.

Consider a system of n bodies (each body may itself be a compound system) with masses m_i and position vectors $\mathbf{q}_i$ locating the center of mass of each body. We introduce a new set of position vectors $\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_{n-1}$ and $\mathbf{Q}$ according to

$$\mathbf{r}_i=(\mu_i/m)^{1/2}[\mathbf{q}_{i+1}-\mathbf{Q}_i], \quad i=1, 2, \cdots n-1,$$
$$\mathbf{Q}=\mathbf{Q}_n, \tag{2.2}$$

where

$$\mathbf{Q}_j=\Big(\sum_{i=1}^{j} m_i\mathbf{q}_i\Big)\Big/\sum_{i=1}^{j} m_i, \quad j=1, 2, \cdots n, \tag{2.3}$$

and

$$(\mu_j)^{-1}=(m_{j+1})^{-1}+\Big(\sum_{i=1}^{j} m_j\Big)^{-1} \quad j=1, 2, \cdots n-1. \tag{2.4}$$

Here m is some conveniently chosen standard mass. In the following unless otherwise stated we make a choice of units, and of m, such that $\hbar^2=2m=1$. The kinetic energy operator for the n bodies in their center-of-mass system then takes the simple form

$$K=-\sum_{i=1}^{n-1}\nabla_{r_i}{}^2. \tag{2.5}$$

A further notational simplification is achieved by representing the set of vectors $\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_{n-1}$ by a single vector $\mathbf{r}=(x_1, x_2, \cdots x_{3(n-1)})$ in a $3(n-1)$ dimensional space with

$$r^2=\sum_{i=1}^{3(n-1)} x_i{}^2. \tag{2.6}$$

If $\mathbf{k}$ is a wave number vector in this space then we have the plane-wave solution

$$(K-k^2)\exp(i\mathbf{k}\cdot\mathbf{r})=0. \tag{2.7}$$

[9] Ideas along these lines have been discussed recently by W. M. MacDonald, University of Maryland Technical Report No. 337, November 1963 (unpublished).

The spherically symmetric solutions of Eq. (2.7) will also be required. They are given by

$$U^{(\pm)}(k,r;p)=C_\pm(k,p)r^{-p/2}H_{p/2}{}^{(1,2)}(kr), \quad p=3n-5, \tag{2.8}$$

where $H_{p/2}{}^{(1)}$ and $H_{p/2}{}^{(2)}$ are cylindrical Hankel functions of the first and second kind, respectively. The normalization factors $C_\pm$ are specified in Eqs. (A11) and (A13). We note the asymptotic behavior

$$H_{p/2}{}^{(1,2)}(kr)\underset{r\to\infty}{\longrightarrow}\left(\frac{2}{kr\pi}\right)^{1/2}\times\exp\left\{\pm i\left[kr-\left(\frac{p+1}{4}\right)\pi\right]\right\}. \tag{2.9}$$

Now suppose each body is actually a bound system of p_i particles; when isolated each system is described by a normalized eigenfunction $\chi_i(\boldsymbol{\varrho}_{i1}, \boldsymbol{\varrho}_{i2}, \cdots \boldsymbol{\varrho}_{ip_i})$ with eigenenergy $-\epsilon_i$. (If the ith system is a single particle, then $\chi_i=1$ and $\epsilon_i=0$.) The $\boldsymbol{\varrho}_{ij}$ are position vectors with respect to the center of mass of the ith system. The totality of these vectors can be represented by a hyperspace vector $\boldsymbol{\varrho}$. The wave function for the n-body system (consisting of

$$\sum_{i=1}^{n} p_i$$

particles) is of the form

$$\Big(\prod_{i=1}^{n_\alpha}\chi_{i\alpha}\Big)\exp(i\mathbf{k}_\alpha\cdot\mathbf{r}_\alpha)\equiv X_\alpha(\boldsymbol{\varrho}_\alpha)\exp(i\mathbf{k}_\alpha\cdot\mathbf{r}_\alpha) \equiv\Phi_\alpha(\boldsymbol{\varrho}_\alpha,\mathbf{r}_\alpha), \tag{2.10}$$

where we have now included a channel index α. Φ_α satisfies the Schrödinger equation

$$(H-V_\alpha-E)\Phi_\alpha=0, \tag{2.11}$$

where

$$E=-\epsilon_\alpha+k_\alpha{}^2, \quad \epsilon_\alpha=\sum_{i=1}^{n_\alpha}\epsilon_{i\alpha}. \tag{2.12}$$

The channel index α indicates the quantum state of the system as well as the particular decomposition into n_α subsystems. V_α is the set of interactions which would actually exist among these systems for r_α finite. The asymptotic form of the wave function can now be written as

$$\Psi_\beta{}^{(\pm)}\to\Phi_\beta(\boldsymbol{\varrho}_\beta,\mathbf{r}_\beta)+T_{\alpha\beta}{}^{(\pm)}(\hat{r}_\alpha,\hat{k}_\beta)X_\alpha(\boldsymbol{\varrho}_\alpha)U_\alpha{}^{(\pm)}(\mathbf{r}_\alpha) \tag{2.13}$$

for $r_\alpha\to\infty$, $\rho_\alpha<\infty$, the carets denoting unit vectors. The scattering problem resolves itself into a determination of the amplitudes $T_{\alpha\beta}$.

With these preliminaries disposed of we are in position to derive the generalized Kato identity. To this end consider the expression

$$J=(\Psi_\alpha{}^{(-)}, [H-E]\Psi_{\beta t}{}^{(+)}) -(\Psi_{\beta t}{}^{(+)}, [H-E]\Psi_\alpha{}^{(-)})^*, \tag{2.14}$$

where $\Psi_{\beta t}^{(+)}$ is a trial function which has the same asymptotic form as $\Psi_\beta^{(+)}$ except that the amplitudes $T_{\alpha\beta}^{(+)}$ are replaced by trial amplitudes $T_{\alpha\beta t}^{(+)}$ (the functions X_α are assumed to be known). Now J, which represents the lack of Hermiticity of H under the assumed boundary conditions, may be evaluated in two ways. We have, of course,

$$J=(\Psi_\alpha^{(-)},[H-E]\Psi_{\beta t}^{(+)}).$$

Alternatively, a multidimensional form of Green's theorem may be used to express J as an integral over a surface at infinity. In fact it is clear from the preceding discussion that the total center-of-mass kinetic energy operator can be written as $-\nabla^2$, where ∇^2 is a multidimensional Laplacian, so that

$$J=-\int_S(\Psi_\alpha^{(-)*}\boldsymbol{\nabla}\Psi_{\beta t}^{(+)}-\Psi_{\beta t}^{(+)}\boldsymbol{\nabla}\Psi_\alpha^{(-)*})\cdot d\mathbf{S}. \quad (2.15)$$

This integral may be evaluated with the aid of Eq. (2.13). We first observe that terms involving products of X_γ^* and X_δ, where γ and δ are two channels which correspond to different groupings of particles into subsystems, do not contribute since there will be no surface element at infinity on which X_γ^* and X_δ overlap. Stated in another way, we may say that as a necessary condition for a term proportional to $X_\gamma^*X_\delta$ to contribute, channels γ and δ must be such that they can be connected by a direct reaction (for which $n_\gamma=n_\delta$) or a breakup reaction (for which $n_\gamma\neq n_\delta$) but not a rearrangement reaction. More detailed considerations show that in fact only terms for which $n_\gamma=n_\delta$ can contribute. This analysis, which we now sketch, is greatly simplified by use of the identity

$$\int_S\Big[\exp(i\mathbf{k}\cdot\mathbf{r})\frac{\partial}{\partial r}U^{(\pm)}(k,r;p)-U^{(\pm)}(k,r;p)\times\frac{\partial}{\partial r}\exp(i\mathbf{k}\cdot\mathbf{r})\Big]f(\hat{r})dS\underset{r\to\infty}{\longrightarrow}f(\mp\hat{k}), \quad (2.16)$$

where the integral is taken over a hypersphere of radius r. This relation, a direct generalization of its three-dimensional counterpart given by Dirac,[10] is derived in the Appendix. The essential point is that due to the rapid oscillations of the integrand there is no contribution unless the coefficient of r appearing in the exponent is zero.

Now consider the class of terms obtained by replacing $\Psi_\alpha^{(-)*}$ in Eq. (2.15) by Φ_α^*, with $\Psi_{\beta t}^{(+)}$ replaced by $T_{\alpha\beta t}X_\gamma U_\gamma^{(+)}$. It will be shown that such terms do not contribute unless $n_\gamma=n_\alpha$. Thus, if we first assume that $n_\alpha<n_\gamma$ we see that since $X_\alpha(\varrho_\alpha)$ vanishes for large ρ_α the surface integral becomes an integral over all coordinates except r_α, which is fixed and large. Furthermore, the error made in replacing r_γ by r_α in the function $U_\gamma^{(+)}$ will be $O(r_\alpha^{-1})$ relative to the leading term. This leading term itself vanishes as r_α tends to infinity, since it contains a finite factor, evaluated with the aid of Eq. (2.16), multiplied by an attenuation factor which, according to Eqs. (2.8) and (2.9), is just $r_\alpha^{-3[n_\gamma-n_\alpha]/2}$. For $n_\alpha=n_\gamma$ the contribution is $-T_{\alpha\beta t}^{(+)}(\hat{k}_\alpha,\hat{k}_\beta)$, where we have used the orthonormality property of the functions X_α. Finally, if $n_\alpha>n_\gamma$ the integral is taken over the surface $r_\gamma=$ constant. Now we may write

$$\exp(i\mathbf{k}_\alpha\cdot\mathbf{r}_\alpha)=\exp(i\mathbf{k}_{\alpha1}\cdot\mathbf{r}_\gamma)\exp(i\mathbf{k}_{\alpha2}\cdot\boldsymbol{\varrho}_\gamma), \quad (2.17)$$

where

$$E_\alpha=-\epsilon_\alpha+(k_{\alpha1}^2+k_{\alpha2}^2),$$
$$E_\gamma=-\epsilon_\gamma+k_\gamma^2=E_\alpha. \quad (2.18)$$

Since $\epsilon_\alpha<\epsilon_\gamma$ (channel α is, so to speak, obtained from channel γ by a breakup reaction), while $k_{\alpha2}^2>0$, we have

$$k_{\alpha2}^2-\epsilon_\alpha>-\epsilon_\gamma \quad (2.19)$$

so that $k_{\alpha1}^2<k_\gamma^2$. Consequently, the equality $\mathbf{k}_{\alpha1}\cdot\mathbf{r}_\gamma=k_\gamma r_\gamma$ cannot be satisfied in the domain of integration and the surface integral vanishes due to the rapid oscillations of the integrand for $r_\gamma\to\infty$. In a similar way the contribution obtained by replacing $\Psi_\alpha^{(-)*}$ by its scattered part and $\Psi_{\beta t}^{(+)}$ by its plane-wave part reduces to $T_{\beta\alpha}^{(-)*}(-\hat{k}_\beta,-\hat{k}_\alpha)$. By extension of this analysis it is easy to see that these are the only contributions so that

$$T_{\beta\alpha}^{(-)*}(-\hat{k}_\beta,-\hat{k}_\alpha)-T_{\alpha\beta t}^{(+)}(\hat{k}_\alpha,\hat{k}_\beta)=(\Psi_\alpha^{(-)},[H-E]\Psi_{\beta t}^{(+)}). \quad (2.20)$$

As a first application of Eq. (2.20) we set $\Psi_{\beta t}^{(+)}=\Psi_\beta^{(+)}$, which leads directly to the well-known reciprocity relation[11]

$$T_{\beta\alpha}^{(-)*}(-\hat{k}_\beta,-\hat{k}_\alpha)=T_{\alpha\beta}^{(+)}(\hat{k}_\alpha,\hat{k}_\beta). \quad (2.21)$$

It can now be verified that the amplitudes $T_{\alpha\beta}$ as defined by Eq. (2.13) are identical to the usual T-matrix elements.[11] We set $\Psi_{\beta t}^{(+)}=\Phi_\beta$ in Eq. (2.20), so that $T_{\alpha\beta t}^{(+)}=0$, and find, with the aid of Eq. (2.21), that

$$T_{\alpha\beta}^{(+)}(\hat{k}_\alpha,\hat{k}_\beta)=(\Psi_\alpha^{(-)},V_\beta\Phi_\beta)=(\Phi_\alpha,V_\alpha\Psi_\beta^{(+)}). \quad (2.22)$$

With the introduction of the error function $\Omega_\alpha^{(\pm)}=\Psi_{\alpha t}^{(\pm)}-\Psi_\alpha^{(\pm)}$, Eq. (2.20) takes the form we have been seeking, that is,

$$T_{\alpha\beta}^{(+)}(\hat{k}_\alpha,\hat{k}_\beta)=T_{\alpha\beta t}^{(+)}(\hat{k}_\alpha,\hat{k}_\beta)+(\Psi_{\alpha t}^{(-)},[H-E]\Psi_{\beta t}^{(+)})-(\Omega_\alpha^{(-)},[H-E]\Omega_\beta^{(+)}). \quad (2.23)$$

A variational principle for $T_{\alpha\beta}$ is obtained by dropping the second-order term, $(\Omega_\alpha^{(-)},[H-E]\Omega_\beta^{(+)})$, in Eq. (2.23). An integral (Schwinger-type) formulation of the variational principle for rearrangement collisions has

[10] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), 3rd ed., p. 191.

[11] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1963).

been given previously by Lippmann.[12] The above differential (Kohn-type[13]) formulation should prove to be more tractable in practical applications. Furthermore, it is more general than earlier multichannel formulations of the variational principle[6,12,13] in that no restriction to two-body channels is made. No essential difficulties are encountered when the effects of spin and the Pauli principle are included.

3. TWO-BODY SCATTERING

The identity given in Eq. (2.23) is a useful starting point for a number of investigations. The construction of minimum principles by bounding the error term in a version of the identity in which the integrals are real has been discussed previously.[6] We wish to show here that interesting relations can be derived by setting the trial function equal to the exact solution of an appropriately chosen problem. Consider, for example, a single-channel two-body scattering problem. We write the central potential as $V(r)=V_0(r)+V_1(r)$ where $V_0(r)$ is a solvable potential, and choose for the trial function $\Psi_t(\mathbf{r})$ the exact solution $\Psi_0(\mathbf{r})$ of

$$(H_0-E)\Psi_0(\mathbf{r})\equiv(-\nabla^2+V_0(r)-E)\Psi_0(\mathbf{r})=0. \quad (3.1)$$

If the scattering amplitude for this base problem is denoted by $f_0(\mathbf{k}_f,\mathbf{k}_i)$, we have

$$T_{\alpha\beta t}{}^{(+)}(\hat{k}_\alpha,\hat{k}_\beta)=-4\pi f_0(\mathbf{k}_f,\mathbf{k}_i). \quad (3.2)$$

To find an expression for the error function in terms of solutions to the base problem we introduce the operator

$$G(z)=(z-H)^{-1} \quad (3.3)$$

and write the operator relation

$$\Psi^{(\pm)}=\Psi_0{}^{(\pm)}+G(E\pm i\eta)V_1\Psi_0{}^{(\pm)}, \quad (3.4)$$

where $\langle\mathbf{r}|\Psi^{(\pm)}\rangle=\Psi^{(\pm)}(\mathbf{r})$ is the correct wave function. With G_0 defined as

$$G_0(z)=(z-H_0)^{-1}, \quad (3.5)$$

it follows that

$$G=G_0+G_0V_1G, \quad (3.6)$$

which is a convenient basis for perturbation theory if V_1 is weak enough. From Eq. (3.4) we have

$$\Omega^{(\pm)}=\Psi_0{}^{(\pm)}-\Psi^{(\pm)}=-G(E\pm i\eta)V_1\Psi_0{}^{(\pm)}, \quad (3.7)$$

so that

$$\begin{aligned}(\Omega_f{}^{(-)},\,&[H-E]\Omega_i{}^{(+)})\\ &=-\{G(E-i\eta)V_1\Psi_{0f}{}^{(-)},\,[H-E]\Psi_{0i}{}^{(+)}\}\\ &=-(\Psi_{0f}{}^{(-)},\,V_1G(E+i\eta)V_1\Psi_{0i}{}^{(+)}),\end{aligned} \quad (3.8)$$

where we have used the relation $G^\dagger(z)=G(z^*)$. A scattering operator T_1 is now defined as

$$T_1{}^{(\pm)}(E)=V_1+V_1G(E\pm i\eta)V_1, \quad (3.9)$$

so that the identity [Eq. (2.23)] becomes

$$\begin{aligned}-4\pi f(\mathbf{k}_f,\mathbf{k}_i)=&-4\pi f_0(\mathbf{k}_f,\mathbf{k}_i)\\ &+(\Psi_{0f}{}^{(-)},\,T_1{}^{(+)}\Psi_{0i}{}^{(+)}).\end{aligned} \quad (3.10)$$

A distorted-wave Born expansion is obtained by iterating Eq. (3.6) for G and combining that expansion with Eqs. (3.9) and (3.10).

To illustrate the utility of Eq. (3.10) we consider, in the remainder of this section, two particular choices for the potential V_0. Suppose V supports N bound states. It is assumed, in this first example, that each bound-state wave function χ_i is known exactly. The condition we place on V_0 is that it be solvable, and that it produce the same bound states as V, so that

$$V_0\chi_i=V\chi_i, \quad i=1, 2, \cdots N. \quad (3.11)$$

It is easily verified that if V_0 is taken as the sum of separable potentials

$$V_0=\sum_{i,j=1}^{N} V|\chi_i\rangle(\mathbf{V}^{-1})_{ij}\langle\chi_j|V, \quad (3.12)$$

where $\mathbf{V}$ is an $N\times N$ matrix with elements

$$V_{ij}=\langle\chi_i|V|\chi_j\rangle, \quad (3.13)$$

then Eq. (3.11) is satisfied. The scattering problem generated by V_0 is easily solved and we just state the solution here. We introduce the function

$$g_i(\mathbf{k})=\langle\mathbf{k}|V|\chi_i\rangle \quad (3.14)$$

and define the $N\times N$ matrices $\mathbf{N}^{(\pm)}(E)$ and $\mathbf{D}(E)$ according to

$$\begin{aligned}[\mathbf{N}^{(\pm)}(E)]_{ij}{}^{-1}=&\int\frac{d^3k}{(2\pi)^3}\frac{g_i(\mathbf{k})g_j(\mathbf{k})}{(k^2+\epsilon_i)(k^2-E\mp i\eta)},\\ D_{ij}(E)=&(E+\epsilon_i)\delta_{ij}.\end{aligned} \quad (3.15)$$

The operator $T_0{}^{(\pm)}$, which satisfies

$$T_0{}^{(\pm)}(E)=V_0+V_0G_0(E\pm i\eta)V_0, \quad (3.16)$$

may then be expressed as

$$\begin{aligned}&\langle\mathbf{k}|T_0{}^{(\pm)}(E)|\mathbf{k}'\rangle\\ &\quad=\sum_{i,j=1}^{N} g_i(\mathbf{k})[\mathbf{N}^{(\pm)}(E)\mathbf{D}^{-1}(E)]_{ij}g_j(\mathbf{k}').\end{aligned} \quad (3.17)$$

This representation exhibits the expected analytic properties of the amplitude. In particular, the poles agree, in position and residue, with those of the true amplitude. One can then express f_0, Ψ_0, and G_0, which enter in Eqs. (3.6) and (3.10), in terms of $T_0{}^{(\pm)}(E)$; these well-known details are omitted here. Since the bound-state poles are included in f_0, the convergence properties of the distorted-wave Born expansion of the residual amplitude [see Eq. (3.10)] is expected to be

[12] B. A. Lippmann, Phys. Rev. **102**, 264 (1956).
[13] W. Kohn, Phys. Rev. **74**, 1763 (1948).

substantially improved compared to the usual Born series.

The fact that V and V_0 generate identical bound states has the following interesting consequence. It is clear that the error function, $\Omega=\Psi_0-\Psi$, is orthogonal to each of the χ_i (note that V_0 is Hermitian). Therefore, at zero energy the error term in the basic identity, Eq. (2.23), is of known sign,[14] i.e.,

$$(\Omega,H\Omega)\gtrless 0, \quad E=0. \tag{3.18}$$

Thus, with the above choice of V_0, the distorted-wave Born approximation, obtained from Eq. (3.10) by replacing T_1 with V_1, gives an upper bound on the scattering length A defined by

$$A=-\lim_{E\to 0} f(\mathbf{k}_f,\mathbf{k}_i). \tag{3.19}$$

This may be viewed as the generalization of the previously obtained result[15] that the ordinary Born approximation provides an upper bound on the scattering length if no bound states exist.

Upper bounds on scattering lengths can be obtained even if the bound-state wave functions are not known exactly. This follows from a theorem[16] which states that if N bound states exist and if $\mathbf{H}$ is an $N\times N$ matrix with elements

$$H_{ij}=\langle\chi_{it}|H|\chi_{jt}\rangle, \tag{3.20}$$

then the operator

$$H_1=H-\sum_{i,j=1}^{N} H|\chi_{it}\rangle(\mathbf{H}^{-1})_{ij}\langle\chi_{jt}|H \tag{3.21}$$

is nonnegative on the space of functions which vanish (or at most go like a constant[14]) at infinity. Here the linearly independent set of trial bound-state functions must be chosen such that $\mathbf{H}$ is negative definite. Taking the expectation value of H_1 with respect to the error function, and use of the zero-energy form of Eq. (2.21), leads to the desired bound. Scattering by compound systems can be treated in a similar way.[6]

The theorem just quoted can be of help in the present problem. If, as is usually the case, the bound-state wave functions are not known exactly the choice $V_0=H-H_1$ is suggested. The scattering problem thus generated has the solution

$$\langle\mathbf{k}|T_0^{(\pm)}(E)|\mathbf{k}'\rangle = \sum_{i,j=1}^{N} h_i(\mathbf{k})[\mathbf{H}+\mathbf{M}^{(\pm)}(E)]_{ij}^{-1}h_j(\mathbf{k}'), \tag{3.22}$$

with

$$h_i(\mathbf{k})=\langle\mathbf{k}|H|\chi_{it}\rangle,$$

$$M_{ij}^{(\pm)}(E)=\int\frac{d^3k}{(2\pi)^3}\frac{h_i(\mathbf{k})h_j(\mathbf{k})}{k^2-E\mp i\eta}. \tag{3.23}$$

It will no longer be true, in general, that f_0 correctly reproduces the bound-state poles of f. Nevertheless, we can be assured that the residual potential V_1 can not support a bound state, since $K+V_1=H_1$ is positive definite. As we have already mentioned, this does not guarantee that the modified Born expansion will converge, since the presence of positive energy resonances as well as bound states can destroy the convergence of the Born series. The method outlined above "subtracts off" only the bound states. This is expected to suffice over a broad range of energies sufficiently removed from resonance regions.

4. MULTICHANNEL SCATTERING

The discussion in Sec. 3 can be taken over directly to construct a distorted-wave Born expansion for the general scattering problem treated in Sec. 2. The basic identity may be written as

$$T_{\alpha\beta}^{(+)}=T_{0\alpha\beta}^{(+)}+(\Psi_{0\alpha}^{(-)},T_1^{(+)}(E)\Psi_{0\beta}^{(+)}), \tag{4.1}$$

where $\Psi_{0\beta}^{(\pm)}$ is defined as the solution of

$$(H_0-E)\Psi_{0\beta}^{(\pm)}=0, \tag{4.2}$$

and satisfies boundary conditions given by Eq. (2.13) with $T_{\alpha\beta}$ replaced by $T_{0\alpha\beta}$. Again, the residual potential V_1 is defined by

$$H=H_0+V_1=K+V_0+V_1 \tag{4.3}$$

and $T_1^{(+)}(E)$ is given formally by Eqs. (3.9) and (3.3). V_0 should be chosen so that V_1 is relieved of the burden of reproducing any of the bound states, either of the total system, or of subsystems in entrance or exit channels; i.e., these bound states should appear in the base problem. Clearly, the construction of a solvable base problem is a more formidable task in the many-body case. Rather than attempting a general discussion of this point at the present time, we consider two special cases which should illustrate the utility of this approach.

We consider first a three-body scattering problem. The problem is greatly simplified if the three particles are taken to be identical. This case is treated here; the case where the particles are not identical will be discussed in a future report.[17]

In Ref. 17 we have established integral equations for three-body scattering amplitudes which have the feature that the kernels involve two-body scattering operators rather than two-body potentials. Iteration of these integral equations leads to the multiple scattering expansions discussed by Watson[18] and others. When the three particles are identical the integral equations are particularly simple. Before quoting them we introduce some notation. Suppose the two-body potential V supports one bound state, with momentum-space wave function $\tilde{\chi}(\mathbf{k})$ and binding energy ϵ. The entrance channel is chosen to be such that two particles

[14] The fact that Ω goes as a constant for large r, rather than vanishes does not alter this conclusion. This point has been discussed in detail in Ref. 15.

[15] L. Spruch and L. Rosenberg, Phys. Rev. **116**, 1034 (1959).

[16] L. Rosenberg, L. Spruch, and T. F. O'Malley, Phys. Rev. **118**, 184 (1960).

[17] L. Rosenberg (to be published).

[18] K. M. Watson, Phys. Rev. **105**, 1388 (1957).

are bound and the third particle has momentum $\mathbf{k}_i$ in the center-of-mass frame (we let $\hbar^2=2m=1$, where m is the mass of each particle.) The exit channel is similarly chosen; the free particle has momentum $\mathbf{k}_f$ with $k_f^2=k_i^2$. The scattering amplitude corresponding to total energy $E=\frac{3}{2}k_i^2-\epsilon$ is written as

$$\mathcal{T}(\mathbf{k}_f,\mathbf{k}_i;E)=B(\mathbf{k}_f,\mathbf{k}_i;E)+\tau(\mathbf{k}_f,\mathbf{k}_i;E). \quad (4.4)$$

Here the Born term is

$$B(\mathbf{k}_f,\mathbf{k}_i;E)=g(\mathbf{k}_f+\tfrac{1}{2}\mathbf{k}_i)\tilde{\chi}(\mathbf{k}_i+\tfrac{1}{2}\mathbf{k}_f) =\frac{g(\mathbf{k}_f+\frac{1}{2}\mathbf{k}_i)g(\mathbf{k}_i+\frac{1}{2}\mathbf{k}_f)}{E-(\mathbf{k}_i+\mathbf{k}_f)^2-k_i^2-k_f^2}, \quad (4.5)$$

with $g(\mathbf{k})$ defined by Eq. (3.14). The second form of Eq. (4.5) is to be used to extend the Born amplitude off the energy shell. The amplitude τ is defined in terms of the two-body scattering operator $T(E)$ which itself satisfies

$$T(E)=V+V(E+i\eta-K)^{-1}T(E). \quad (4.6)$$

In the impulse approximation the elastic amplitude is

$$I(\mathbf{k}_f,\mathbf{k}_i;E)=2\int\frac{d^3k}{(2\pi)^3}\tilde{\chi}(\tfrac{1}{2}\mathbf{k}_f+\mathbf{k}) \times\langle\mathbf{k}_f+\tfrac{1}{2}\mathbf{k}|\,T(E-\tfrac{3}{2}k^2)\,|\,\mathbf{k}_i+\tfrac{1}{2}\mathbf{k}\rangle\tilde{\chi}(\tfrac{1}{2}\mathbf{k}_i+\mathbf{k}) \quad (4.7)$$

and the breakup amplitude [three particles free in the final state, with momenta $\mathbf{k}_f$, $\mathbf{k}_f'$, and $-(\mathbf{k}_f+\mathbf{k}_f')$] is

$$I(\mathbf{k}_f,\mathbf{k}_f';\mathbf{k}_i;E)=2\tilde{\chi}(\tfrac{1}{2}\mathbf{k}_i+\mathbf{k}_f') \times\langle\mathbf{k}_f+\tfrac{1}{2}\mathbf{k}_f'|\,T(E-\tfrac{3}{2}k_f'^2)\,|\,\mathbf{k}_i+\tfrac{1}{2}\mathbf{k}_f'\rangle. \quad (4.8)$$

The time-reversed amplitude $I(\mathbf{k}_f;\mathbf{k}_i,\mathbf{k}_i';E)$ (three particles free in the initial state) is obtained from Eq. (4.8) with the aid of the reciprocity relation, Eq. (2.21). The integral equations determining τ in Eq. (4.4) can now be written as

$$\tau(\mathbf{k}_f,\mathbf{k}_i;E) =I(\mathbf{k}_f,\mathbf{k}_i;E)+\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}I(\mathbf{k}_f;\mathbf{k}',\mathbf{k};E) \times[E+i\eta-(\mathbf{k}+\mathbf{k}')^2-k^2-k'^2]^{-1} \times\tau(\mathbf{k},\mathbf{k}';\mathbf{k}_i;E), \quad (4.9)$$

and

$$\tau(\mathbf{k}_f,\mathbf{k}_f';\mathbf{k}_i;E) =I(\mathbf{k}_f,\mathbf{k}_f';\mathbf{k}_i;E)+2\int\frac{d^3k}{(2\pi)^3} \times\langle\mathbf{k}_f+\tfrac{1}{2}\mathbf{k}_f'|\,T(E-\tfrac{3}{2}k_f'^2)\,|\,\mathbf{k}+\tfrac{1}{2}\mathbf{k}_f'\rangle \times[E+i\eta-(\mathbf{k}+\mathbf{k}_f')^2-k^2-k_f'^2]^{-1} \times\tau(\mathbf{k}_f',\mathbf{k};\mathbf{k}_i;E). \quad (4.10)$$

As in the case of the Born amplitude, $\tau(\mathbf{k}_f,\mathbf{k}_i;E)$ may be continued off the energy shell by writing

$$\tilde{\chi}(\tfrac{1}{2}\mathbf{k}_f+\mathbf{k}')=\frac{g(\frac{1}{2}\mathbf{k}_f+\mathbf{k}')}{E-(\mathbf{k}'+\mathbf{k}_f)^2-k'^2-k_f^2} \quad (4.11)$$

in Eq. (4.9) and varying k_f^2 with E fixed.

Having set up the equations which determine the exact amplitude we now introduce a base problem by replacing V with

$$V_0=\frac{V|\chi\rangle\langle\chi|\,V}{\langle\chi|\,V|\chi\rangle}, \quad (4.12)$$

which is Eq. (3.12) specialized to the case $N=1$. According to Eq. (3.17) the two-body scattering operator $T_0^{(+)}(E)$ is given by[19]

$$\langle\mathbf{k}|\,T_0^{(+)}(E)\,|\,\mathbf{k}'\rangle=-\frac{1}{2}\frac{g(\mathbf{k})g(\mathbf{k}')}{E+\epsilon}S^{(+)}(E) \quad (4.13)$$

with

$$[S^{(+)}(E)]^{-1}=\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{g^2(\mathbf{k})}{(2k^2-E-i\eta)(2k^2+\epsilon)}. \quad (4.14)$$

We then see that

$$I_0(\mathbf{k}_f,\mathbf{k}_i;E) =\int\frac{d^3k}{(2\pi)^3}B(\mathbf{k}_f,\mathbf{k};E)\frac{S^{(+)}(E-\frac{3}{2}k^2)}{E-\frac{3}{2}k^2+\epsilon}B(\mathbf{k},\mathbf{k}_i;E)$$

and

$$\tau_0(\mathbf{k}_f,\mathbf{k}_i;E) =I_0(\mathbf{k}_f,\mathbf{k}_i;E)+\int\frac{d^3k}{(2\pi)^3}B(\mathbf{k}_f,\mathbf{k};E) \times\frac{S^{(+)}(E-\frac{3}{2}k^2)}{E-\frac{3}{2}k^2+\epsilon}\tau_0(\mathbf{k},\mathbf{k}_i;E). \quad (4.15)$$

Finally, the amplitude $\mathcal{T}_0(\mathbf{k}_f,\mathbf{k}_i;E)$ becomes, for this choice of separable potential,

$$\mathcal{T}_0(\mathbf{k}_f,\mathbf{k}_i;E) =B(\mathbf{k}_f,\mathbf{k}_i;E)+\int\frac{d^3k}{(2\pi)^3}B(\mathbf{k}_f,\mathbf{k};E) \times\frac{S^{(+)}(E-\frac{3}{2}k^2)}{E-\frac{3}{2}k^2+\epsilon}\mathcal{T}_0(\mathbf{k},\mathbf{k}_i;E). \quad (4.16)$$

This solution may be verified by comparing Eqs. (4.16) and (4.15), each iterated once, and making use of the defining relation Eq. (4.4). With the elastic amplitude known the inelastic amplitudes can be obtained directly; no other integral equations need be solved. We find that

[19] We have changed our notation and units somewhat in this section in order to facilitate subsequent comparison with results of Ref. 7.

the breakup amplitude is given by

$$\mathcal{T}_0(\mathbf{k}_f,\mathbf{k}_f';\mathbf{k}_i;E)$$
$$=\tfrac{1}{4}\mathcal{S}\left[g(\mathbf{k}_f+\tfrac{1}{2}\mathbf{k}_f')\frac{S^{(+)}(E-\frac{3}{2}k_f'^2)}{E-\frac{3}{2}k_f'^2+\epsilon}\mathcal{T}_0(\mathbf{k}_f',\mathbf{k}_i;E)\right], \quad (4.17)$$

where $\mathcal{S}$ indicates a sum over all permutations of final momenta. The amplitude $\mathcal{T}(\mathbf{k}_f,\mathbf{k}_f',\mathbf{k}_i,\mathbf{k}_i';E)$, which describes the collision in which the three particles are free in initial and final states, can be decomposed into a sum of disconnected parts, each describing a two-body collision with the third particle unaffected, and the remaining connected part $\mathcal{T}^C$ which contains the essential three-body aspect of the problem. This connected amplitude reduces, in the base problem introduced above, to

$$\mathcal{T}_0{}^C(\mathbf{k}_f,\mathbf{k}_f',\mathbf{k}_i\mathbf{k}_i';E)$$
$$=\frac{1}{12}\mathcal{S}\left[g(\mathbf{k}_f+\tfrac{1}{2}\mathbf{k}_f')\frac{S^{(+)}(E-\frac{3}{2}k_f'^2)}{E-\frac{3}{2}k_f'^2+\epsilon}\mathcal{T}_0(\mathbf{k}_f';\mathbf{k}_i,\mathbf{k}_i';E)\right]. \quad (4.18)$$

Diagrammatic representations of all these equations can easily be constructed. Such diagrams appear in Ref. 7, as well as in an earlier paper which dealt with the construction of a unitary impulse approximation.[20] The similarity between the present model and the unitary impulse approximation will be discussed in more detail in the future.[17]

The integral equation for $\mathcal{T}_0$ is no more difficult to solve than the Lippmann-Schwinger equation for the two-body scattering amplitude. Eq. (4.16) has been derived previously by Amado[7] who used techniques not based on ordinary potential scattering theory. The virtue of the present derivation is twofold. Firstly, the connection between Amado's techniques and ordinary potential theory is exhibited; we have produced the potential which generates Amado's model. Secondly, systematic corrections to this first approximation can be obtained from Eq. (4.1) using a (presumably convergent) distorted-wave Born expansion. Generalizations of the model to deal with certain stripping amplitudes have been suggested by Amado.[7] These generalizations can also be treated by the techniques described here.

It may be of interest to observe that the solvable model discussed by Skornyakov and Ter-Martirosyan,[21] in which three identical particles interact by means of two-body zero-range potentials, appears as a special case of the base problem discussed above. We need ony replace $g(\mathbf{k})$ by its zero-range limit, which is just a constant. This is equivalent to replacing $\chi(\mathbf{r})$ by its asymptotic form.

As another illustration of these techniques we consider a deuteron stripping reaction $X(d,f)Y$, where X is the ground state of a target nucleus. Particle c in the deuteron is captured by X to form Y, the nucleon f acting as a "spectator." The basic assumption made[8] is that in a pure stripping reaction the process depends solely on the interaction V_{cf} between particles c and f, and then only when c is outside the nucleus X. That is, let the total potential $V=V_{Xf}+V_{cf}+V_{cX}$ be replaced by

$$V_0=V_{Xf}+V_{cf}[\epsilon(r_0-r_c)+\epsilon(r_c-R)]+V_{cX}, \quad (4.19)$$

where r_c is the separation of c and X, $\epsilon(x)$ is the step function

$$\begin{aligned}\epsilon(x)&=0, \quad x<0\\ &=1, \quad x>0,\end{aligned} \quad (4.20)$$

and r_0 is the radius of nucleus X. The term $V_{cf}\epsilon(r_c-R)$ is inserted so that for $r_c>R$ (with $R\gg r_0$) the potential is switched on, allowing the formation of the deuteron in the entrance channel. The statement of the stripping approximation is then

$$T_{0\alpha\beta}\approx 0, \quad (4.21)$$

where $T_{0\alpha\beta}$ is the stripping amplitude associated with the potential V_0. Since all the bound states in entrance and exit channels are present in the base problem as well as the true problem, it is reasonable to suppose that the distorted-wave Born expansion about the residual potential $V_1=V-V_0$ is convergent. If we keep only the first term in the expansion the stripping amplitude $T_{\alpha\beta}$ becomes

$$\begin{aligned}T_{\alpha\beta}&\approx T_{0\alpha\beta}+(\Psi_{0\alpha}{}^{(-)},[V_{cf}\epsilon(r_c-r_0)\epsilon(R-r_c)]\Psi_{0\beta}{}^{(+)})\\ &\approx(\Psi_{0\alpha}{}^{(-)},[V_{cf}\epsilon(r_c-r_0)]\Psi_{0\beta}{}^{(+)}),\end{aligned} \quad (4.22)$$

where we have used Eq. (4.21) and have taken R large enough so that $\epsilon(R-r_c)$ is effectively unity. The distorted waves $\Psi_{0\alpha}{}^{(-)}$ and $\Psi_{0\beta}{}^{(+)}$ can be computed with the aid of elastic and inelastic scattering data (with stripping ignored) for the processes d on X and f on Y.

While the result expressed by Eq. (4.22) is well known, the above derivation has, in addition to its simplicity, two distinct advantages. Firstly, the variational nature of the result is displayed, since the term neglected in the first line of Eq. (4.22) is of second order as can be seen by comparison with Eq. (2.23). This formal property, along with the expectation that the errors $\Psi_\alpha{}^{(-)}-\Psi_{0\alpha}{}^{(-)}$ and $\Psi_\beta{}^{(+)}-\Psi_{0\beta}{}^{(+)}$ will in fact be small in a pure stripping reaction, is perhaps the explanation for the "surprising" success of Eq. (4.22) in practical applications. Another virtue of the present derivation is that it indicates a systematic method of improving the approximation or testing its validity by looking at additional terms in the presumably covergent distorted-wave Born expansion.

[20] L. Rosenberg, Phys. Rev. **131**, 874 (1963).

[21] G. V. Skornyakov and K. A. Ter-Martirosyan, Zh. Eksperim. i Teor. Fiz. **31**, 775 (1957) [English transl.: Soviet Phys.—JETP **4**, 648 (1957)]; see also, L. D. Faddeev, *ibid.* **39**, 1459 (1960) [English transl.: *ibid.* **12**, 1014 (1961)].

APPENDIX

We shall derive Eq. (2.16) of the text; we first consider the function $U^{(+)}$, given by Eq. (2.8), and replace $H^{(1)}$ by its asymptotic form given by Eq. (2.9). The normalization constant C_+ in Eq. (2.8) is specified below.

To treat the surface integral we introduce (following Sommerfeld[22]) a set of hyperspherical coordinates defined in terms of the Cartesian coordinates x_i, by

$$\begin{aligned} x_1&=r\cos\theta \\ x_2&=r\sin\theta\cos\varphi_1 \\ x_3&=r\sin\theta\sin\varphi_1\cos\varphi_2 \\ &\vdots \\ x_{p+1}&=r\sin\theta\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{p-1}\cos\varphi_p \\ x_{p+2}&=r\sin\theta\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{p-1}\sin\varphi_p, \end{aligned} \tag{A1}$$

where $p+2$ is the dimensionality of the space. The orientation of the coordinate system is chosen such that $\mathbf{k}$ is along the x_1 axis. The surface integral in Eq. (2.16) then becomes

$$C_+\left(\frac{2}{k\pi}\right)^{1/2}\exp\{-i(p+1)\pi/4\}\int d\omega_\varphi\int_0^\pi \sin^p\theta d\theta r^{p+1} \times\left(\exp\{ikr\cos\theta\}\frac{\partial}{\partial r}\left[\frac{1}{r^{(p+1)/2}}\exp(ikr)\right] -\frac{1}{r^{(p+1)/2}}\exp(ikr)\frac{\partial}{\partial r}(\exp\{ikr\cos\theta\})\right)f(\hat{r}), \tag{A2}$$

with

$$\int d\omega_\varphi=\int_0^\pi \sin^{p-1}\varphi_1 d\varphi_1\int_0^\pi \sin^{p-2}\varphi_2 d\varphi_2\cdots \times\int_0^\pi \sin\varphi_{p-1}d\varphi_{p-1}\int_{-\pi}^{\pi} d\varphi_p. \tag{A3}$$

We first examine the integral over θ,

$$I_\theta\equiv\int_0^\pi \sin^p\theta d\theta r^{p+1}(\ \ )f(\hat{r}), \tag{A4}$$

which reduces to

$$I_\theta=ikr^{p+1}r^{-(p+1/2)}\int_0^\pi \sin^p\theta d\theta\exp\{ikr(1+\cos\theta)\} \times(1-\cos\theta)f(\hat{r})[1+O(1/r)]. \tag{A5}$$

We now write

$$\begin{aligned} v&=kr(1+\cos\theta), \\ f(\hat{r})&=g(v/kr) \end{aligned} \tag{A6}$$

(suppressing the dependence of g on the angle variables φ_i) so that

$$I_\theta=ik^{-(p-1)/2}\int_0^{2kr} v^{(p-1)/2}e^{iv}(2-v/kr)^{(p+1)/2} \times g(v/kr)dv+O(1/r). \tag{A7}$$

By applying the mean value theorem to each of the r-dependent factors in the integrand we see that we may let $r\to\infty$ in the integrand, the error being $O(1/r)$. Therefore, in the limit of infinite r, I_θ becomes

$$\begin{aligned} I_\theta&=2^{(p+1)/2}g(0)ik^{-[(p-1)/2]}\int_0^\infty v^{(p-1)/2}e^{iv}dv \\ &=f(-\hat{k})2i\left(\frac{2}{k}\right)^{(p-1)/2}i^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right). \end{aligned} \tag{A8}$$

Thus Eq. (2.16), with the upper sign chosen, is verified provided C_+ is chosen such that

$$C_+(2/k\pi)^{1/2}(-i)^{(p+1)/2}\Omega_\varphi 2i(2/k)^{(p-1)/2} \times\Gamma[(p+1)/2]=1, \tag{A9}$$

where[22]

$$\Omega_\varphi=\int d\omega_\varphi=2\pi\frac{(p+1)/2}{\Gamma[(p+1)/2]}. \tag{A10}$$

This simplifies to

$$C_+=\pi^{1/2}(k/2)^{p/2}[2\pi i(p+1)]^{-1}. \tag{A11}$$

(The significance of this normalization is that $U^{(+)}(k,r;\hat{p})$ satisfies the unit source condition, i.e.,

$$\int\frac{\partial U^{(+)}}{\partial r}d\sigma=1, \tag{A12}$$

where the integration is taken over a sphere of radius $r\to 0$.) Equation (2.16) can be verified for the lower sign by choosing

$$C_-=-\pi^{1/2}(k/2)^{p/2}[2\pi i(p+1)]^{-1}=C_+^* \tag{A13}$$

so that $U^{(+)*}=U^{(-)}$. If we now take the complex conjugate of Eq. (2.16), with the upper sign chosen, and transform $\mathbf{r}$ to $-\mathbf{r}$ in the integrand we obtain the stated result, which completes the proof.

[22] A. Sommerfeld, *Partial Differential Equations* (Academic Press Inc., New York, 1949), p. 227.